2024 Maths Challenge

Intermediate

Solutions for Students

AUSTRALIAN MATHS TRUST

MATHS CHALLENGE

Intermediate Challenge — Solutions for Students

I1 Pointy Numbers

a Since the sum of three consecutive digits is three times the middle digit, that sum is always divisible by 3.

Alternative i

In a 7-digit pointy number, the sum of its first three digits is divisible by 3 and the sum of its last three digits is divisible by 3. So the sum of all its digits is divisible by 3 if and only if its middle digit is divisible by 3. Since the first digit and middle digit differ by 3, the first digit must also be divisible by 3.

Alternative ii

In a 7-digit pointy number, the sum of its 2nd, 3rd, and 4th digits is divisible by 3 and the sum of its 5th, 6th, and 7th digits is divisible by 3. So the sum of all its digits is divisible by 3 if and only if its first digit is divisible by 3.

Alternative iii

A 7-digit pointy number has the form $a, a \pm 1, a \pm 2, a \pm 3, a \pm 2, a \pm 1, a$. So the sum of its digits is $7a \pm 9$. Hence the pointy number is divisible by 3 if and only if 3 divides $7a$. This means its first digit a must be divisible by 3.

Alternative iv

List all 7-digit pointy numbers and note the multiples of 3:

$$1234321, 2345432, \underline{3210123}, \underline{3456543}, 4321234, 4567654,$$

$$5432345, 5678765, \underline{6543456}, \underline{6789876}, 7654567, 8765678, \underline{9876789}.$$

The multiples of 3 are underlined. These can be found using a calculator or the rule for division by 3. The first digit of each of these is divisible by 3.

b A number is divisible by 6 if and only if it is both even and divisible by 3.

Alternative i

For a pointy number to be even, its last (and therefore also first) digit must be even and not 0. We list all even pointy numbers systematically in increasing order, noting that every pointy number must have an odd number of digits:

$$212, 232, 434, 454, 656, 676, 878, 898,$$
$$\underline{21012}, 23432, 43234, \underline{45654}, 65456, 67876, \underline{87678},$$
$$2345432, 4321234, 4567654, \underline{6543456}, \underline{6789876}, 8765678,$$
$$234565432, 432101234, 456787654, 654323456, 876545678,$$
$$23456765432, \underline{45678987654}, 65432123456, \underline{87654345678},$$
$$2345678765432, \underline{6543210123456}, 8765432345678,$$
$$234567898765432, 876543212345678,$$
$$\underline{87654321012345678}.$$

The multiples of 3, which can be checked via their digit sum, are underlined. So in total there are nine pointy numbers that are divisible by 6.

Alternative ii

A 3-digit pointy number has the form

$$a, a \pm 1, a.$$

The sum of the digits is $3a \pm 1$. Neither $3a + 1$ nor $3a - 1$ can be divisible by 3. So no 3-digit pointy number is divisible by 6.

A 5-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $5a \pm 4$. Checking positive even values of a, we find $5a + 4$ is a multiple of 3 only for $a = 4$, and $5a - 4$ is a multiple of 3 only for $a = 2$ or $a = 8$. So the only 5-digit pointy numbers divisible by 6 are 21012, 45654, and 87678.

A 7-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $7a \pm 9$, which is a multiple of 3 only for $a = 6$. So the only 7-digit pointy numbers divisible by 6 are 6543456 and 6789876.

A 9-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $9a \pm 16$. Since 16 is not divisible by 3, neither $9a + 16$ nor $9a - 16$ can be divisible by 3. So no 9-digit pointy number is divisible by 6.

An 11-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 5, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $11a \pm 25$. Checking positive even values of a, we find $11a + 25$ is a multiple of 3 only for $a = 4$, and $11a - 25$ is a multiple of 3 and positive only for $a = 8$. So the only 11-digit pointy numbers divisible by 6 are 45678987654 and 87654345678.

A 13-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 5, a \pm 6, a \pm 5, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $13a \pm 36$, which is a multiple of 3 only for $a = 6$. This is too big for an upward pointy number, but works for a downward pointy number. So the only 13-digit pointy number divisible by 6 is 6543210123456.

A 15-digit pointy number has the form

$$a, a \pm 1, \ldots, a \pm 6, a \pm 7, a \pm 6, \ldots, a \pm 1, a.$$

So the sum of the digits is $15a \pm 49$. Since 49 is not divisible by 3, neither $15a + 49$ nor $15a - 49$ can be divisible by 3. So no 15-digit pointy number is divisible by 6.

A 17-digit pointy number has the form

$$a, a \pm 1, \ldots, a \pm 7, a \pm 8, a \pm 7, \ldots, a \pm 1, a.$$

So the sum of the digits is $17a \pm 64$. Checking positive even values of a, we find $17a + 64$ is a multiple of 3 only for $a = 4$ (which is too big), and $17a - 64$ is a multiple of 3 only for $a = 2$ (which is too small) or $a = 8$. So the only 17-digit pointy number divisible by 6 is 87654321012345678.

Finally, the only 19-digit pointy number is 9876543210123456789, which is not even and so not divisible by 6.

So in total there are nine pointy numbers that are divisible by 6.

c First note that each pair of same-placed digits in the upward and downward pointy numbers will have the same sum. Call that common sum n.

Because the two pointy numbers have the same number of digits, the sum of the two pointy numbers is

$$n + 10n + 100n + \cdots = n(1 + 10 + 100 + \cdots).$$

Since the first (and therefore the last) digit of each pointy number must be at least 1, the value of n must be at least 2. Since pointy numbers have at least three digits, the sum in brackets is at least 111. Thus the sum of the two pointy numbers is not prime.

d First note that each pair of corresponding digits in the 7-digit upward and downward pointy numbers will have the same sum. Call that common sum n. The first digit of a 7-digit downward pointy number is at least 3 and first digit of a 7-digit upward pointy number is at least 1. So n is at least 4.

When two 7-digit numbers are added, the result is either a 7-digit sum or an 8-digit sum starting with 1.

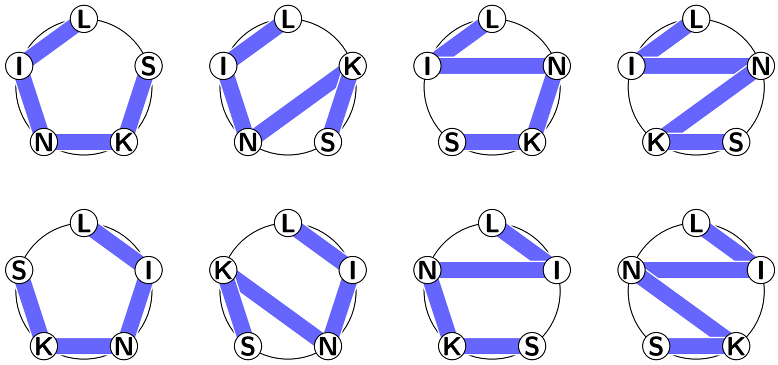
If the sum is a 7-digit palindrome, then it has the form $nnnnnnn$.
If $n = 4$, there is exactly one pair of pointy numbers: $\{1234321, 3210123\}$.
If $n = 5$, there are exactly two pairs of pointy numbers: $\{1234321, 4321234\}$, $\{2345432, 3210123\}$.
Similarly, for $n = 6, 7, 8, 9$, there are exactly 3, 4, 5, 6 pairs of pointy numbers respectively.

If the sum of the two 7-digit pointy numbers starts with 1 and is a palindrome, then it ends with 1. So $n = 11$. In this case there are exactly five pairs of pointy numbers:
$\{2345432, 9876789\}, \{3456543, 8765678\}, \{4567654, 7654567\}, \{5678765, 6543456\}, \{6789876, 5432345\}$.

So the number of required pairs of 7-digit pointy numbers is $1 + 2 + 3 + 4 + 5 + 6 + 5 = 26$.

I4 Word Links

a



These arrangements can be found by systematically listing all possibilities using the following argument. If the letter I is not placed adjacent to the letter L, then the link between them would separate one of the remaining letters from the other two and hence that link would eventually need to be crossed. Hence the I must be adjacent to L. Similarly, the N must be placed adjacent to I or L, but not between them. Thus there are four ways of placing L, I, N. For each of these arrangements, there are two ways of placing K and S. So we get the eight crossless arrangements above.

b Alternative i

Working clockwise from the letter L, no matter where it is placed, there are 4 letters to choose from to fill the next position. For each of those choices, there are 3 letters to choose from to fill the next position, then 2 for the next position, leaving only 1 letter for the last position. So the number of ways of placing I, N, K, S is $4 \times 3 \times 2 \times 1 = 24$. From Part **a**, 8 of these arrangements are crossless. So the probability that a random arrangement of the five letters in the word LINKS is crossless is $\frac{8}{24} = \frac{1}{3}$.

Alternative ii

After placing letter L, the letter I must be placed adjacent to it to avoid a crossing. The probability of this is $\frac{2}{4}$. The letter N must then be placed in one of two positions to avoid a crossing. The probability of this is $\frac{2}{3}$. Then the letter K can be placed in either of the two remaining positions and S in the last available position. So the probability that a random arrangement of the five letters in the word LINKS is crossless is $\frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$.

c **Alternative i**

The first letter in a six letter word can be placed in any of 6 positions. The second letter must be adjacent to it to avoid a crossing. So there are 2 positions to choose from for the second letter. For each of these choices, there are only 2 positions to choose from for the third letter to avoid a crossing. Similarly, there are then only 2 choices for each of the fourth and fifth letters. So the number of crossless arrangements is $6 \times 2 \times 2 \times 2 \times 2 = 96$.

Alternative ii

Regardless of where the first letter is placed, a six letter arrangement is crossless if and only if the fifth letter is adjacent to the sixth letter and the arrangement for the first five letters is crossless. From Part **a**, there are exactly 8 five letter crossless arrangements for each position of the first letter. So the number of six letter crossless arrangements is $6 \times 2 \times 8 = 96$.

d All arrangements of two letter and three letter words are crossless, so their probability is 1. With the first letter fixed, exactly 4 of the 6 arrangements of a four letter word are crossless, so their probability is $2/3 > 0.01$. From Part **b**, the probability that a five letter arrangement is crossless is $1/3 > 0.01$.

Alternative i

Relative to the first letter, the number of arrangements of the remaining 5 letters in a six letter word is $5 \times 4 \times 3 \times 2 \times 1 = 120$. So, from the first solution of Part **c**, the probability that a random arrangement of the six letters in a word is crossless is $2^4/120 = 2/15 > 1/100$.

By a similar argument we have the following probabilities.
The probability that a random arrangement of the seven letters in a word is crossless is
$2^5/(6 \times 5 \times 4 \times 3 \times 2 \times 1) = 2/45 > 1/100$.
The probability that a random arrangement of the eight letters in a word is crossless is
$2^6/(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 4/315 > 1/100$.
The probability that a random arrangement of the nine letters in a word is crossless is
$2^7/(8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 1/315 < 1/100$.

So the least number of letters for which the probability that a random arrangement is crossless is below 0.01 is nine.

Alternative ii

The first letter in an n-letter word can be placed in any of n positions. The second letter can then be placed in any of $n - 1$ positions, and so on. So the number of arrangements for the n letters is $1 \times 2 \times 3 \times \cdots \times n = n!$.

Generalising the first solution of Part **c**, we see that the number of crossless arrangements in an n-letter word is $n2^{n-2}$. So the probability of an arrangement being crossless is $P_n = n2^{n-2}/n! = 2^{n-2}/(n-1)!$. Thus $P_6 = 16/120 = 2/15 > 0.01$, $P_7 = 32/720 = 2/45 > 0.01$, $P_8 = 2^6/7! = 4/315 > 0.01$, and $P_9 = 2^7/8! = 1/315 < 0.01$.

Hence the least number of letters for which the probability that a random arrangement is crossless is below 0.01 is nine.