2024 Maths Challenge

Junior

Solutions for Students

AUSTRALIAN MATHS TRUST

MATHS CHALLENGE

Junior Challenge — Solutions for Students

J1 Pointy Numbers

a For a number to be divisible by 5, its last digit must be 5 or 0. The first and last digits of a pointy number are the same and 0 is not allowed as a first digit. So all pointy numbers that are divisible by 5 start with 5. The largest upward pointy number that starts with 5 is 567898765. The largest downward number that starts with 5 is 54321012345. Therefore the largest pointy number that is divisible by 5 is 54321012345.

b Since the sum of three consecutive digits is three times the middle digit, that sum is always divisible by 3.

Alternative i

In a 7-digit pointy number, the sum of its first three digits is divisible by 3 and the sum of its last three digits is divisible by 3. So the sum of all its digits is divisible by 3 if and only if its middle digit is divisible by 3. Since the first digit and middle digit differ by 3, the first digit must also be divisible by 3.

Alternative ii

In a 7-digit pointy number, the sum of its 2nd, 3rd, and 4th digits is divisible by 3 and the sum of its 5th, 6th, and 7th digits is divisible by 3. So the sum of all its digits is divisible by 3 if and only if its first digit is divisible by 3.

Alternative iii

A 7-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of its digits is $7a \pm 9$. Hence the pointy number is divisible by 3 if and only if 3 divides $7a$. This means its first digit a must be divisible by 3.

Alternative iv

List all 7-digit pointy numbers and note the multiples of 3:

$$1234321, 2345432, \underline{3210123}, \underline{3456543}, 4321234, 4567654,$$

$$5432345, 5678765, \underline{6543456}, \underline{6789876}, 7654567, 8765678, \underline{9876789}.$$

The multiples of 3 are underlined. These can be found using a calculator or the rule for division by 3. The first digit of each of these is divisible by 3.

c A number is divisible by 6 if and only if it is both even and divisible by 3.

Alternative i

For a pointy number to be even, its last (and therefore also first) digit must be even and not 0. We list all even pointy numbers systematically in increasing order, noting that every pointy number must have an odd number of digits:

$$212, 232, 434, 454, 656, 676, 878, 898,$$

$$\underline{21012}, 23432, 43234, \underline{45654}, 65456, 67876, \underline{87678},$$

$$2345432, 4321234, 4567654, \underline{6543456}, \underline{6789876}, 8765678,$$

$$234565432, 432101234, 456787654, 654323456, 876545678,$$

$$23456765432, \underline{45678987654}, 65432123456, \underline{87654345678},$$

$$2345678765432, \underline{6543210123456}, 8765432345678,$$

$$234567898765432, 876543212345678,$$

$$\underline{87654321012345678}.$$

The multiples of 3, which can be checked via their digit sum, are underlined. So in total there are nine pointy numbers that are divisible by 6.

Alternative ii

A 3-digit pointy number has the form

$$a, a \pm 1, a.$$

The sum of the digits is $3a \pm 1$. Neither $3a + 1$ nor $3a - 1$ can be divisible by 3. So no 3-digit pointy number is divisible by 6.

A 5-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $5a \pm 4$. Checking positive even values of a, we find $5a + 4$ is a multiple of 3 only for $a = 4$, and $5a - 4$ is a multiple of 3 only for $a = 2$ or $a = 8$. So the only 5-digit pointy numbers divisible by 6 are 21012, 45654, and 87678.

A 7-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $7a \pm 9$, which is a multiple of 3 only for $a = 6$. So the only 7-digit pointy numbers divisible by 6 are 6543456 and 6789876.

A 9-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $9a \pm 16$. Since 16 is not divisible by 3, neither $9a + 16$ nor $9a - 16$ can be divisible by 3. So no 9-digit pointy number is divisible by 6.

An 11-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 5, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $11a \pm 25$. Checking positive even values of a, we find $11a + 25$ is a multiple of 3 only for $a = 4$, and $11a - 25$ is a multiple of 3 and positive only for $a = 8$. So the only 11-digit pointy numbers divisible by 6 are 45678987654 and 87654345678.

A 13-digit pointy number has the form

$$a, a \pm 1, a \pm 2, a \pm 3, a \pm 4, a \pm 5, a \pm 6, a \pm 5, a \pm 4, a \pm 3, a \pm 2, a \pm 1, a.$$

So the sum of the digits is $13a \pm 36$, which is a multiple of 3 only for $a = 6$. This is too big for an upward pointy number, but works for a downward pointy number. So the only 13-digit pointy number divisible by 6 is 6543210123456.

A 15-digit pointy number has the form

$$a, a \pm 1, \ldots, a \pm 6, a \pm 7, a \pm 6, \ldots, a \pm 1, a.$$

So the sum of the digits is $15a \pm 49$. Since 49 is not divisble by 3, neither $15a + 49$ nor $15a - 49$ can be divisible by 3. So no 15-digit pointy number is divisible by 6.

A 17-digit pointy number has the form

$$a, a \pm 1, \ldots, a \pm 7, a \pm 8, a \pm 7, \ldots, a \pm 1, a.$$

So the sum of the digits is $17a \pm 64$. Checking positive even values of a, we find $17a + 64$ is a multiple of 3 only for $a = 4$ (which is too big), and $17a - 64$ is a multiple of 3 only for $a = 2$ (which is too small) or $a = 8$. So the only 17-digit pointy number divisible by 6 is 87654321012345678.

Finally, the only 19-digit pointy number is
9876543210123456789, which is not even and so not divisible by 6.

So in total there are 9 pointy numbers that are divisible by 6.

d First note that each pair of same-placed digits in the upward and downward pointy numbers will have the same sum. Call that common sum n.

Because the two pointy numbers have the same number of digits, the sum of the two pointy numbers is

$$n + 10n + 100n + \cdots = n(1 + 10 + 100 + \cdots).$$

Since the first (and therefore the last) digit of each pointy number must be at least 1, the value of n must be at least 2. Since pointy numbers have at least three digits, the sum in brackets is at least 111. Thus the sum of the two pointy numbers is not prime.

J4 Card Fractions

a Alternative i

We use a common denominator. Since $\frac{4}{8} = \frac{20}{40}$, $\frac{5}{8} = \frac{25}{40}$, $\frac{4}{5} = \frac{32}{40}$, we have $\frac{4}{8} < \frac{5}{8} < \frac{4}{5}$.

Alternative ii

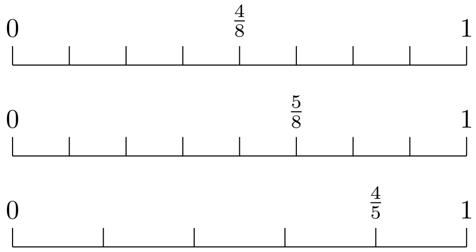
Converting each fraction to a decimal, we have $\frac{4}{5} = 0.8$, $\frac{4}{8} = 0.5$, $\frac{5}{8} = 0.625$. Thus $\frac{4}{8} < \frac{5}{8} < \frac{4}{5}$.

Alternative iii

We know that $\frac{4}{8} < \frac{5}{8}$ because these fractions have the same denominator and $4 < 5$. Since $\frac{5}{8} < \frac{6}{8} = \frac{3}{4}$ and $\frac{4}{5} = \frac{16}{20} > \frac{15}{20} = \frac{3}{4}$, we have $\frac{5}{8} < \frac{4}{5}$. So $\frac{4}{8} < \frac{5}{8} < \frac{4}{5}$.

Alternative iv

We use number lines.



Thus $\frac{4}{8} < \frac{5}{8} < \frac{4}{5}$.

b The cards Andy flipped cannot be labelled 1 since 1 is still visible.

If the cards Andy flipped were labelled 2, the fractions would have values $\frac{1}{6}, \frac{2}{6} = \frac{1}{3}, \frac{1}{2}$, which are indeed ordered from smallest to largest.

If the cards Andy flipped were labelled with a number greater than 2, the middle fraction would be at least $\frac{3}{6} = \frac{1}{2}$, while the fraction on the right would be at most $\frac{1}{3}$, hence not greater than the middle fraction.

So the only possible number on the cards Andy flipped is 2.

c Each fraction must be less than 1. So we know from the fraction on the right that the number on the A cards must be less than 7, and from the middle fraction that the number on the B cards must be greater than 7.

The table below summarises the possible outcomes for different A and B numbers. In each case, the fractions can be compared by any of the methods described in the solutions to Part **a**.

B	A	Comparison	Outcome
8	6	$\frac{7}{8} > \frac{6}{7}$	A too small.
9	6	$\frac{7}{9} < \frac{6}{7}$	Possible.
9	$\leqslant 5$	$\frac{7}{9} > \frac{A}{7}$	A too small.
10	6	$\frac{7}{10} < \frac{6}{7}$	Possible.
10	5	$\frac{7}{10} < \frac{5}{7}$	Possible.
10	$\leqslant 4$	$\frac{7}{10} > \frac{A}{7}$	A too small.

So the only possible combinations of numbers on the cards labelled A and B are respectively: 6 and 9, 6 and 10, 5 and 10.

d Alternative i

Since 1 is the smallest number chosen and 2024 is the largest number chosen, we know $\frac{1}{2024}$ must be the smallest fraction. So writing M for the cards with the mystery number, the fractions must be ordered in the following way:

$$\frac{\boxed{1}}{\boxed{2024}} < \frac{\boxed{\mathrm{M}}}{\boxed{2024}} < \frac{\boxed{1}}{\boxed{\mathrm{M}}}$$

As the value for the cards labelled M increases, the middle fraction gets larger while the last fraction gets smaller. In particular, by testing different values for M, we can find that when M is 44, the middle fraction is $\frac{44}{2024} = \frac{1}{46}$, which is smaller than the last fraction, $\frac{1}{44}$, while when M is 45, the middle fraction is $\frac{45}{2024}$, which is larger than the last fraction, $\frac{1}{45} = \frac{45}{2025}$. So the possible values for Andy's unknown number are $2, 3, 4, \ldots, 44$, meaning it could be any of 43 possible values.

Alternative ii

Simplify the inequality $\frac{\mathrm{M}}{2024} < \frac{1}{\mathrm{M}}$ by multiplying both sides by 2024M. Since the multiplier is positive, we find $\mathrm{M}^2 < 2024$, so $\mathrm{M} < \sqrt{2024}$, which is slightly less than $\sqrt{2025} = 45$. So M can take any integer value less than 45 and greater than 1, that is, there are 43 possible values for M.