Maths Enrichment

Euler Student Notes

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AUSTRALIAN MATHS TRUST
MATHS ENRICHMENT

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Chapter 1

Primes and the Euclidean Algorithm

Opening problem

Three friends are playing a game with plastic coins. Arjan has 112 coins in front of him, Bei has 189 coins and Chloe has 144 coins. Whoever has the most coins always takes the next turn (when there is a tie, they play rock–paper–scissors to decide). On their turn, a player chooses someone else's number of coins and discards that many coins from their own pile. For example, Bei goes first and chooses to remove Chloe's number, 144, leaving himself with $189 - 144 = 45$ coins. The winner is the last player with coins left in front of them. How many coins will the winner keep?

You might get an idea of what is happening by playing through a few games. But how can you be sure the same thing will happen every time?

Prime decomposition

Prime numbers are like the atoms of arithmetic – we can think of them as the building blocks of all the other natural numbers (counting numbers).

To see why, let's try to break down the number 120 into smaller parts.

When we write $4 \times 30 = 120$, the numbers 4 and 30 on the left are called *factors* or *divisors* and the resulting number 120 on the right is called their *product*. The goal is to write 120 as a product of factors that are as small as possible.

Like 120 itself, 4 and 30 are examples of *composite numbers* because they can be written as a product of two smaller numbers: $4 = 2 \times 2$ and $30 = 5 \times 6$ (as well as 3×10 or 2×15). Although it is also true that $4 = 1 \times 4$ and $30 = 1 \times 30$, this doesn't help us break the numbers down because one of the factors is the same as the product itself – we only want to use factors that are both smaller than the product. This means we also need to ignore the factor 1, sometimes called the *trivial* factor.

Now we know that $120 = 4 \times 30 = (2 \times 2) \times (5 \times 6)$. The factors 2, 2 and 5 are *prime* because they cannot be broken down any further: their only factors are 1 and themselves. On the other hand, the factor $6 = 2 \times 3$ is composite. So we can write $120 = (2 \times 2) \times (5 \times (2 \times 3))$.

The brackets aren't necessary and the factors can be written in any order. So we have the following *prime decomposition* of 120:

$$120 = 2 \times 2 \times 2 \times 3 \times 5.$$

Chapter 2

Advanced Counting

Opening problem

Whole numbers like 247 and 985 410 are called *one-way numbers*. Their digits are all different and are in either increasing or decreasing order. All single-digit numbers from 0 to 9 can be thought of as one-way numbers by default.

How many one-way numbers are there altogether?

You could probably list all the ones with a very small or very large number of digits. But the others? Forget it! Time to learn some new techniques.

When should I add and when should I multiply?

When counting the number of ways events can take place, systematic listing can be a useful starting point. Sometimes listing a handful of cases is enough to unlock some hidden pattern and the rest can be taken care of with a few quick calculations. This chapter consolidates some of the basics covered in the *Dirichlet Student Notes* and then adds a few more techniques to your counting toolkit.

When should I add? Do this if the problem is an 'OR' situation and you need to split it into two or more separate cases. The word 'separate' here is important. In mathematics we say *mutually exclusive* to mean that there is no overlap between the cases – if one of them happens then the other(s) cannot happen. (We'll cover what to do if there *is* overlap a little later.)

Example 1. In Australia, the format of a car number plate depends on the state or territory you are in. Some end with a letter and some end with a digit. How many possibilities are there for the last symbol?

Solution. The last symbol is either a letter OR a digit. It cannot be both, so these cases are mutually exclusive. There are 26 possibilities if it is a letter and 10 possibilities if it is a digit. So there are $26 + 10 = 36$ possibilities in all.

By adding, all we are doing is avoiding writing out the full list of 26 letters followed by 10 digits.

When should I multiply? Do this if the problem is an 'AND' situation, so two or more independent things are happening at the same time. When we say *independent*, we mean that one choice doesn't affect the other(s).

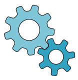
Chapter 3

Graphs and Networks

Opening problem

In a room containing an odd number of people, everyone wants to shake hands with exactly half of the other people.

Is this possible if there are 9 people in the room? What if there are 19 people? What about 29 people?

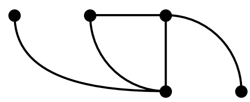


Drawing a diagram is much easier than keeping track of lists of people. But that's only if it works! How do you know when such a diagram can't exist?

Introduction to graphs

One of Leonhard Euler's many contributions was the development of a new type of mathematics called *graph theory*. Never heard of Euler? See the biography at the front of these notes!

Yes, you have heard of graphs before. You use column graphs and sector graphs (pie charts) to represent and compare data. You may have started exploring graphs of equations by plotting points in the number plane. What do these graphs have in common? They are all about representing relationships between different objects: a type of cheese is paired up with the number of people who like it, or an x value is paired up with a y value to form the coordinates of a point.



This chapter is about a different type of graph altogether – but fundamentally it is still about relationships between objects.

The 'objects' in these graphs are called *vertices*. Each vertex is represented by a solid dot, as in the example above. When two vertices are 'related' (whatever that means), we join them by an *edge*. Edges can be straight or curvy, the shape doesn't matter. In fact, there is really only one rule:

> Every edge must have a vertex at both ends.

It's even okay if an edge has the same vertex at each end, in which case it is called a *loop*. We can join the same pair of vertices by more than one edge; we refer to this as a *double edge*, *triple edge*, and so on. Edges are allowed to cross each other, but we only think of them as actually meeting if they share a vertex. Sometimes we try to avoid crossing edges, but that's not always possible. Finally, the rule doesn't say that every vertex must be attached to an edge – if it isn't, we say it is *isolated*.

Chapter 4

Methods of Proof

Opening problem

A graph is called *heterovalent* if all its vertices have different degrees.

(a) Draw a heterovalent graph with 4 vertices.

(b) Prove that a heterovalent graph with 2 or more vertices must have loops or multiple edges.

What goes wrong if you try to avoid loops or multiple edges in your solution to (a)? How can you turn this into a proof of (b) for *any* number of vertices?

Direct proof and contradiction

How do you prove that something is true in mathematics? One way is to start with something you know and use various rules to deduce the thing you want to show. This is called *direct proof*.

Example 1. Let n be an integer. Show that if n is odd, then n^2 is also odd.

This is just a special case of the well-known fact that 'odd times odd equals odd'. But since the exercise is to prove *that it is true, we shouldn't rely on rote-learned facts, especially if we aren't really sure why they work.*

Solution. The thing we *know* is that n is odd. That is, n is 1 more than an even number. All even numbers are multiples of 2. Putting this together, this means we can let $n = 2k + 1$ for some other integer k. We don't need to know *which* integer k happens to be, just that it exists. (We could also let $n = 2k - 1$ or $2k + 3$, among many other possibilities.)

The *rules* that we can use for this problem are our algebra skills. Given that $n = 2k + 1$, we have

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= (2k+1)(2k+1) \\ &= 2k(2k+1) + 1(2k+1) \\ &= 4k^2 + 2k + 2k + 1 \\ &= 4k^2 + 4k + 1. \end{aligned}$$

(If you already know some short cuts for expanding these 'binomial products', you can probably skip a few steps here.)

Chapter 5

Introduction to Diophantine Equations

Opening problem

To send a parcel, I need to make a total postage of \$7.10 from a collection of stamps. Can I do this with 45c and 30c stamps? What about 45c and 20c stamps? Find all possible ways to solve these problems.

Systematic listing via guess-and-check could get us there ... eventually. But even then, how can you be sure you haven't missed a solution?

Diophantus of Alexandria

How many pairs of numbers can you think of that add to 7?

A couple? A handful? Lots and lots? Infinitely many maybe?

The answer to this question all depends on what sort of solutions you're interested in. Does $3+4$ count? Probably. What about $0+7$? Maybe. What about $9+(-2)$? Hmm. And $2\frac{2}{3}+4\frac{1}{3}$ and $-0.2+7.2$ and $\pi+(7-\pi)$ and ... the list is endless if you allow *any* kinds of numbers.

Often a problem only makes sense for integers: positive or negative whole numbers or zero. Sometimes it only makes sense for positive integers, also known as the natural numbers or counting numbers. Sometimes we can allow zero but no negatives. And sometimes there's a natural limit to how big the solutions can get.

The study of these types of problems dates back many centuries. A *Diophantine equation* is an equation where all of the numbers involved, including the coefficients (fixed numbers) and the pronumerals (variables), are integers. This class of problems is named in honour of the ancient Greek mathematician Diophantus of Alexandria, who developed many sophisticated techniques in the 3rd century CE. To this day, this is an important area of mathematical research with many famous unsolved problems.

The search for numbers adding to 7 can be phrased in terms of the equation $x+y=7$. If x and y can be any integers, there are infinitely many solutions: for any integer value of x there is a corresponding integer value of y equal to $7-x$. If they must both be positive, we only get six solutions, which can be written as ordered pairs: $(x,y)=(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$. Maybe we're not so interested in the order of x and y since they can be swapped without changing the original problem: we could write them as the three *un*ordered pairs $\{x,y\}=\{1,6\},\{2,5\},\{3,4\}$.

In this chapter we'll start to explore these types of problems in more detail.

Chapter 6

Chase that Angle

Opening problem

In this cool symmetrical logo, two straight lines meet at an angle of $100°$ at the centre of the circle.

What are the sizes of the angles at the other three corners of the arrowhead?

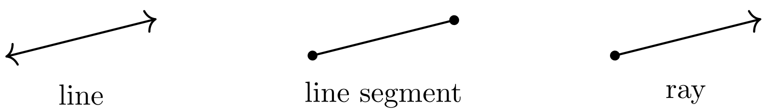
Sure... you *could* just use your trusty protractor. But there's enough hidden information to find the exact answers. The trick is knowing where to look.

The basics

Remember all those geometrical properties you were allowed to forget for graphs and networks in Chapter 3? Time to *un*forget them!

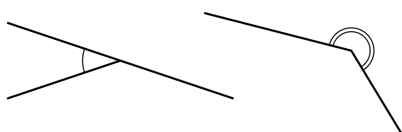
In Euclidean geometry, a line is a ... hang on, exactly what *is* a line? It's a strange thing that we all *know* what we mean by 'line', but writing down *precisely* what we mean is quite a challenge! Euclid himself defined a line as a 'breadthless length'.

Let's agree that a *line* is straight and it stretches off forever in both directions. A 'line' that stops at both ends it is called a *line segment* or *interval*. If it stops (or starts?) at one end and stretches off forever in the other direction, it is called a *ray*.



Sometimes we mark points as solid dots and sometimes we add arrows to distinguish between the different types of objects, as shown above. But often it is clear what we mean without the dots and arrows, so there is no hard-and-fast rule about this.

When two lines, line segments or rays meet at a point, they create an *angle* (or several angles) in between them. Sometimes we mark the angle we're interested in with one or more arcs.



We frequently measure angles in degrees and there are $360°$ in a full revolution. *Why* we use 360 degrees is another story – look up 'Egyptian calendars' and 'Babylonian number system' sometime. (And also 'radians' and 'gradians' if you're really keen!)

Chapter 7

Number Bases

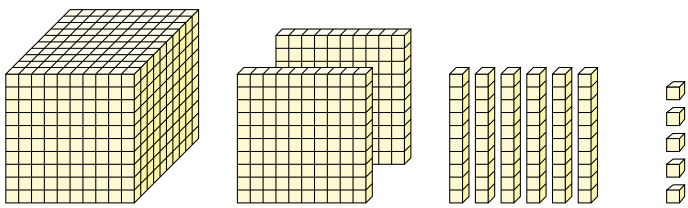
Opening problem

When a three-digit number in base 9 is converted to base 8, each digit increases by 1. Find all such numbers.

Making sense of this question is the hard bit! Time for a reminder that the decimal system is not the only way to represent numbers.

Decimal recap

As you well know, the number 1265 means 1 thousand, plus 2 hundreds, plus 4 tens, plus 7 ones. Let's represent one as an individual cube, ten as a 10×1 strip of cubes, one hundred as a 10×10 'flat' and one thousand as a $10 \times 10 \times 10$ block. Then we can visualise 1265 with the following diagram:



$$1265 = \mathbf{1} \times 10^3 + \mathbf{2} \times 10^2 + \mathbf{6} \times 10 + \mathbf{5} \times 1.$$

This is the decimal number system, otherwise known as 'base 10'. It is likely that a number system based on ten different symbols $(0, 1, 2, \ldots, 9)$ developed because we have five digits (fingers and thumbs) on each hand. The fact we use the word 'digits' in both contexts is a bit of a hint!

Other bases

There is nothing *mathematically* special about the number ten. The same structure can be used in any other base. For example, in base 7 the only available digits are 0, 1, 2, 3, 4, 5 and 6. After that we need to introduce a digit in a new column and reset the 'ones' to zero. Similarly, when we reach 66, we need to introduce yet another column. So the positive integers in base 7 are

$$1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 16, 20, 21, 22, \ldots, 65, 66, 100, 101, \ldots$$

After 6, we say 'one-zero, one-one, one-two, . . . ' instead of 'ten, eleven, twelve, . . . '.

What would '1265' mean in base 7? We definitely don't want to get there by listing.

Chapter 8

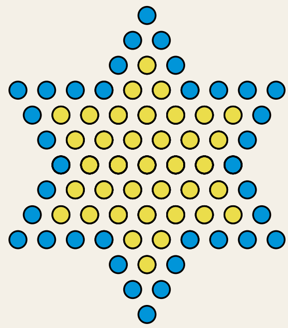
Sequences and Figurate Numbers

Opening problem

This symmetrical 6-pointed star is made up of 36 dark counters on the boundary and 37 lighter counters on the interior.

Carl has a *lot* of time on his hands. He adds 3000 more light counters to the collection and arranges them to make the interior of a much larger pattern of the same type.

How many more dark counters will he need to complete the boundary?



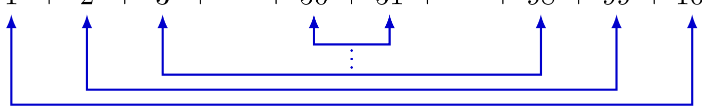
Carl may have time to do this the long way. But there is more than enough structure to work this out without guessing. Time to learn a few new skills!

A clever short cut

The story goes that, before he went on to become one of the most influential thinkers of the 17th and 18th centuries, 7-year-old Carl Friedrich Gauss was challenged, along with his classmates, to add the numbers from 1 to 100. Maybe it was a punishment or just to keep them busy, but most likely it was to see what strategies they could come up with. *What would you try?*

Within a few seconds, Carl announced that he had found the answer: 5050. There are many versions of this tale and most of the details are almost certainly inaccurate. But the short cut is still clever! And surprisingly simple: pair up the numbers from opposite ends. That is, $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$ and so on up to $50 + 51 = 101$. There are 50 pairs adding to 101, so the sum of all the numbers is $50 \times 101 = 5050$. Not bad for a 7-year-old!

$$1 + 2 + 3 + \cdots + 50 + 51 + \cdots + 98 + 99 + 100$$



It's a nice party trick if you ever get challenged to do this type of thing. But is it actually useful in any other situation? Absolutely!

Chapter 9

Modular Arithmetic

Opening problem

What is the last digit of 7^{8^6}? What are the last two digits? What about the last three digits?

This is a number with over 220 000 digits. And yet, it is possible to answer all of these questions with a few calculations by hand! How, you ask?

From 'base' to 'modulo'

If you have worked through Chapter 7 on number bases, you will already know that there are unusual ways to do arithmetic. So a statement like '$5 + 6 = 14$' makes perfect sense in base 7 even though it is nonsense in decimal.

When you practise with number bases enough, the idea that '5 plus 6 makes a ones digit of 4 in base 7' becomes second nature.

In this chapter, we'll make things easier: ignore anything past the ones column, so now you don't even need to worry about the carry!

Instead of '5 plus 6 *has last digit* 4 *in base* 7' we will say '5 plus 6 *is congruent to* 4 *modulo* 7'. The language is a bit different, but the underlying idea is the same. It's all about remainders.

The next few sections recap some of the basics in the 'Clock arithmetic' chapter of the *Dirichlet Student Notes*. (Note that 'congruent' is another name for 'equivalent'.)

Remainders and congruence

If you want to know what day of the week it is in 635 days, you certainly wouldn't 'count on' from today 635 times. The trick is to notice that days reset every week, so we can ignore multiples of 7. In this case, 630 days (90 weeks) can be ignored, so the answer comes from 'counting on' only 5 times. In fact, we could even 'count back' by 2 days instead.

In the language of modular arithmetic, we say that 635 is congruent to 5 modulo 7. It is also congruent to −2 modulo 7. These statements are written as

$$635 \equiv 5 \pmod 7 \qquad \text{and} \qquad 635 \equiv -2 \pmod 7,$$

or just

$$635 \equiv 5 \equiv -2 \pmod 7.$$