Maths Enrichment

Gauss Student Notes

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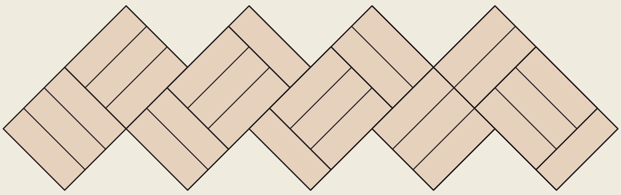
AUSTRALIAN MATHS TRUST | MATHS ENRICHMENT

Contents

Chapter 1
Sequences and Counting

Opening problem

Nine 3×3 squares are arranged to form a zigzag shaped path. The path is then tiled with 3×1 rectangular pavers. One possible tiling pattern is shown.



How many different tiling patterns are there?

There are thousands, so drawing them all is no good. It is better to build up from a smaller problem – and discover a hidden Fibonacci-style sequence!

Arithmetic sequences and series

Before we get to new counting techniques, let's start by recapping some terminology and results from Chapter 8 of the *Euler Student Notes.*

An *arithmetic sequence* is an ordered list of numbers that increase by the same fixed amount, called the *common difference*. For example, $\{1, 4, 7, 10, \ldots\}$ has common difference $d = 3$, and $\{13, 7, 1, -5, \ldots\}$ has common difference $d = -6$ (the 'increase' is negative). Using sequence notation, we can summarise these examples as follows.

A sequence $\{t_1, t_2, t_3, t_4, \ldots\}$ is called *arithmetic* if there exists a fixed number d such that $t_{n+1} = t_n + d$ for all $n \geq 1$.

Provided we also know the first term, often denoted a, we can calculate any other term simply by adding d enough times.

Given an arithmetic sequence $\{t_1, t_2, t_3, t_4, \ldots\}$ with first term a and common difference d, the nth term is $t_n = a + (n - 1)d$.

Example 1. Find the 100th term of the arithmetic sequence $\{1, 4, 7, 10, \ldots\}$.

Solution. $a = 1$ and $d = 3$, so $t_{100} = 1 + 99 \times 3 = 298$.

Chapter 2

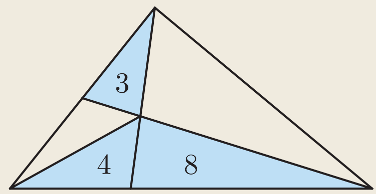
Parallels

Opening problem

Three line segments are drawn in a triangle, dividing it into five smaller triangles.

The areas of three small triangles are shown in the diagram.

Find the total area of the big triangle.



There seems to be something missing here. We don't know any lengths or angles or how the line segments were placed. Is this really enough to go on?

The basic area formula

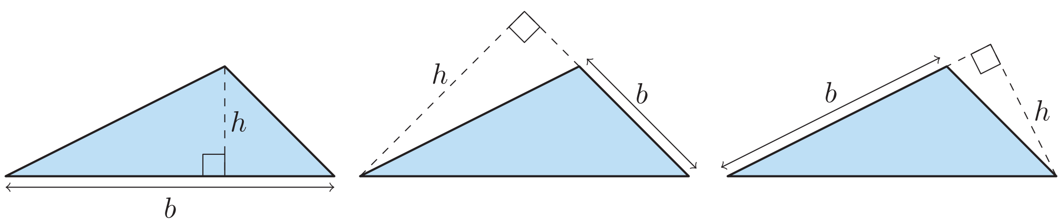
There are lots of different ways to calculate the area of a triangle. Which method you choose all depends on what information you have at hand. If you know the lengths of all three sides, there's a neat method called Heron's formula. If you know two side lengths and the angle between them, trigonometry comes to the rescue. If you can calculate the cross product of three-dimensional vectors ... and so on!

The point is that all these fancy techniques are a consequence of the same basic area formula. Here, *altitude* is just another name for perpendicular height.

Suppose a triangle has base length b and altitude h. Then its area is $\frac{1}{2}bh$.

Where is the base exactly? Always at the 'bottom'? No, any of the three sides can be the base. The only thing that matters is that the height we use in the formula must be perpendicular (at right angles) to the chosen base. When the triangle has an obtuse angle, this means the altitude might need to be measured *outside* the triangle, by extending the side that was chosen as the base.

Here is the same triangle with three different choices of base. The values of b and h are different in each case, but the quantity $\frac{1}{2}bh$ is always the same.



Chapter 3
Prime Decomposition

Opening problem

The number 6 is *perfect*: it equals the sum of its proper factors, $6 = 1 + 2 + 3$. The number 28 is also perfect.

Suppose p is a prime number with the property that $2^p - 1$ is also a prime number. Prove that $2^{p-1}(2^p - 1)$ is a perfect number.

Testing 6 ($p = 2$) and 28 ($p = 3$) is easy enough. Even 496 ($p = 5$) is pretty manageable. But how do you verify it works for *all* possible cases?

Fundamental theorem of arithmetic

I'm thinking of a two-digit number. Including itself and 1, it has exactly 12 different positive factors. That doesn't sound like much to go on – guess-and-check, anyone?

In fact, we have enough information to pinpoint exactly what kind of number we're looking for. It turns out that there are infinitely many numbers with 12 factors (the same is true for any other number of factors). But restricting to two digits quickly narrows the search.

All of this depends on a very important fact about the natural numbers (positive integers). This fact is *so* important we call it the *fundamental theorem of arithmetic*.

FTA: Every integer greater than 1 is either a prime or a product of two or more primes. Moreover, the factors in this prime decomposition are unique.

For example, $98 = 2 \times 7 \times 7 = 2^1 \times 7^2$ is the unique prime decomposition of 98. There is no other way to write 98 as a product of primes (apart from shuffling the order of the factors, which doesn't count as different).

The proof of the FTA is a bit technical and we won't discuss it here. One of the key steps appears as Exercise 15 in Chapter 10.

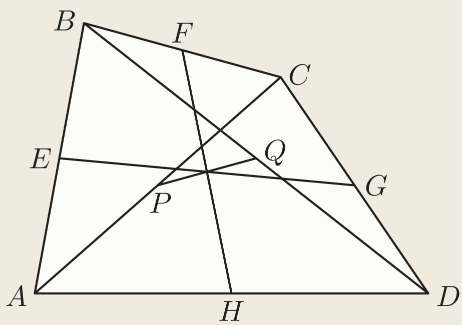
In the meantime, the next section recaps the ideas in Chapter 6 of the *Dirichlet Student Notes*. And we'll look for that two-digit number I was thinking of.

Chapter 4
Congruence

Opening problem

Given a quadrilateral $ABCD$, let E, F, G and H be the midpoints of sides AB, BC, CD and DA, respectively. Also, let P be the midpoint of diagonal AC and Q the midpoint of diagonal BD.

Prove that line segments EG, FH and PQ meet at a common point and that this point bisects each of the three line segments.



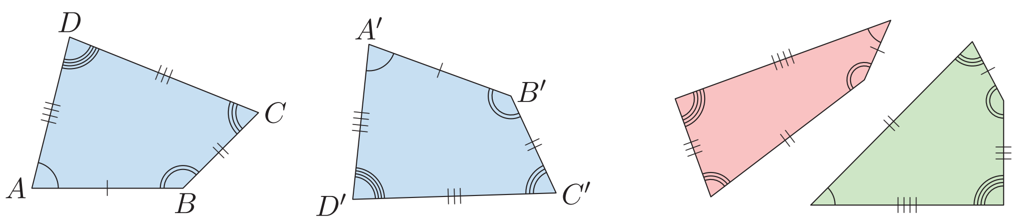
This looks pretty tough, especially as there are two parts to prove – common point *and* bisecting. Actually, the second part is where we should start.

Congruent triangles

Loosely, two shapes are *congruent* if one is an exact copy of the other. Rotating or reflecting the shapes does not make them different.

For polygons, it is enough that the sides and angles can be paired up in such a way that corresponding sides have the same length and corresponding angles are equal. But we also have to keep track of how the sides and angles relate to each other.

The first two quadrilaterals below are congruent, with equal side lengths $AB = A'B'$, $BC = B'C'$ and so on, and equal angles $\angle ABC = \angle A'B'C'$, $\angle BCD = \angle B'C'D'$ and so on. We write $ABCD \cong A'B'C'D'$ for short, being careful to list the vertices in the corresponding order.



The last two quadrilaterals are very unusual. Clearly they 'look different', so they are *non*-congruent. But as marked, all the side lengths and angles *do* happen to pair up, just not in the correct order! So we need to be a little careful with our definition.

Chapter 5
Expanding and Factorising

Opening problem

For each of the following numbers, find at least one prime factor.

(a) 64 027 (b) 99 999 919 (c) 28 331 (d) 54 641

We can use divisibility tests to easily rule out 2, 3, 5 and 11. We could try dividing by $7, 13, 17, \ldots$ but is there a more direct way to get to the answers?

The distributive law

Much of what we'll do in this chapter boils down to applying the following rule, over and over again.

Distributive law: $a(b+c) = ab + ac$ and $(a+b)c = ac + bc$.

The rules work both ways. When we go left to right, we call it 'expanding brackets'. When we go right to left, we call it 'factorising' (we turn it into a product of *factors*).

Even the simple idea of collecting like-terms by adding *coefficients* (the numbers in front) is an application of this rule. For example, $7x - 3x$ can be simplified to $4x$ using the second version of the distributive law:

$$7x - 3x = 7x + (-3)x = (7 + (-3))x = 4x.$$

Of course, we don't do this every time, once we're comfortable using a short cut.

Similarly, more complex ideas such as expanding perfect squares or factorising a difference of cubes (or worse!) can all be done laboriously, one step at a time, with just the basic rule. Developing short cuts will not only speed things up, it will help us uncover some unexpected connections along the way.

The good news is that, if you've worked through the previous chapters, you've come across some of the key ideas already – even if you didn't know it!

Basic expanding and factorising

Applying the distributive law twice gives the following rule for expanding a so-called *binomial product*.

Binomial distributive law: $(a+b)(c+d) = ac + ad + bc + bd$.

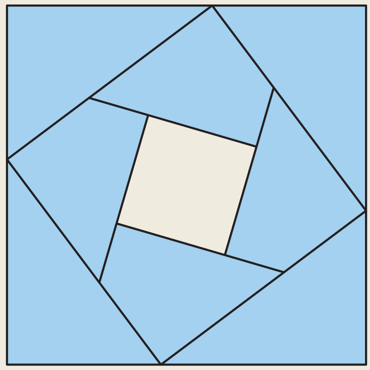
Chapter 6
Similarity

Opening problem

Angelina cuts out eight identical right-angled triangles from a piece of blue card, each with sides of length 30 cm, 40 cm and 50 cm. She arranges them as shown.

The inner four triangles overlap each other, but the outer four triangles do not overlap any of the others.

In square centimetres, what is the area of the uncovered square in the middle?

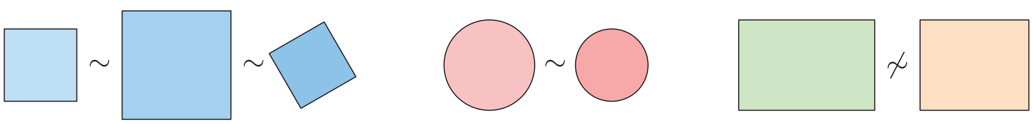


The hidden pieces are crucial to this problem, so redrawing the diagram with everything visible is a good start. Okay … now what?

Centre of dilation and scale factor

Loosely, two shapes are *similar* if one is a uniformly scaled copy of the other. The amount of 'stretch' must be the same in all directions but, like congruence, rotations and reflections are also allowed.

All squares are similar and all circles are similar. Not all rectangles are similar, since their sides might be in different proportions. We use the symbol '$\sim$' as a short cut for 'is similar to'.



Similar polygons have the same sets of angles. In fact, for triangles, this is enough to guarantee similarity – more about this in the next section. On the other hand, the rectangles above illustrate that the angle property alone doesn't guarantee similarity for other types of polygons.

So what is similarity exactly?

Let's approach the definition via a process called *dilation*.

For example, suppose we are given a polygon and we want to create a copy that is the 'same shape' but twice the size.

Chapter 7

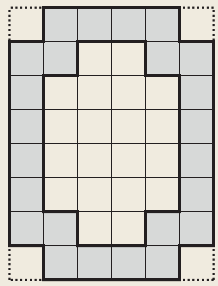
Further Diophantine Equations

Opening problem

The design shown is formed by removing one square from each corner of a rectangular grid and shading a border of thickness 1 unit.

In this 6 by 8 example, there are 24 shaded squares in the border and 20 unshaded squares in the interior.

Ignoring rotations, how many such designs have equal numbers of border and interior squares?



Finding one or two examples? That doesn't sound too tough. But how can you be sure you have them all? Time for some more Diophantine techniques.

Linear Diophantine equations

In the study of *Diophantine equations*, we are only interested in integer solutions. Maybe our job is to find them all, or maybe it is to show that there aren't any.

A *linear* Diophantine equation has the form $ax + by = c$, where a, b and c are fixed integers. We are interested in finding integer values of x and y, if they exist.

If a, b and c are all 'nice', a little guess-and-check can often do the job. Occasionally, we can narrow the search using common factors. We can even use one solution to find many others. Here are two key ideas from Chapter 5 of the *Euler Student Notes*.

Tip: If two out of three terms in a linear Diophantine equation have a common factor, then the third has the same factor.

The 'plus zero' trick: Suppose (x, y) is a solution of $ax + by = c$. Then $(x - bk, y + ak)$ is also a solution for any integer k.

Example 1. Describe infinitely many integer solutions of $5x + 6y = 100$.

Solution. The terms $6y$ and 1000 are both even, so x must be even.

The terms $5x$ and 1000 are both mulitples of 5, so y must be a multiple of 5.

One possible solution is $(x, y) = (8, 10)$. So $(8 - 6k, 10 + 5k)$ is a solution for every integer k. *Check this by substituting* $x = 8 - 6k$, $y = 10 + 5k$ *into* $5x + 6y$.

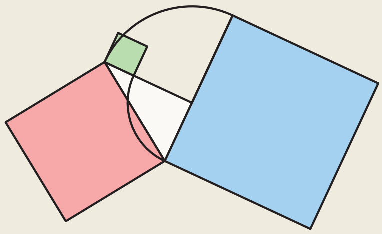
But what if we want to solve something nasty like $183x + 429y = 357$?

Chapter 8
Pythagoras' Theorem

Opening problem

Two quarter circles and three squares are added to the sides of a right-angled triangle as shown.

Find a relationship between the areas of the three squares.



Put away those rulers! What we want is a relationship between areas that works for *any* such arrangement. The quarter circles are crucial – but how?

Equal areas

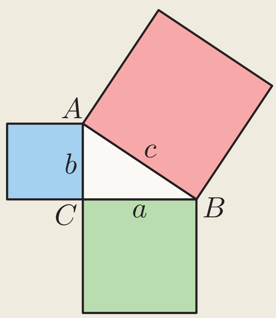
Pythagoras' theorem, also commonly known as *the Pythagorean theorem*, is surely one of the most celebrated results in all of mathematics. The precise history of the theorem is very unclear, but it was almost certainly known to different civilisations centuries before the time of the Ancient Greek mathematician himself, who lived in the 6th century BCE. A quick search online provides lots of interesting reading!

The theorem states that, given any right-angled triangle, the areas of three squares drawn on the sides are related in a very precise way – the larger area is the sum of the smaller two. The surprising thing is that it works both ways. That is, the area condition can *only* happen when there is a right angle.

In the following theorem statement, note the convention of using the same letter to label angles (upper case) and side lengths (lower case) that are opposite each other.

Pythagoras' theorem: Suppose $\triangle ABC$ has sides of length a, b and c.

Then $a^2 + b^2 = c^2$ if and only if $\angle C$ is a right angle.



In a right-angled triangle, the longest side is called the *hypotenuse* (from the Greek for 'stretching under').

Chapter 9
Further Modular Arithmetic

Opening problem

The brilliant mathematician Leonhard Euler once claimed that the equation $a^4 + b^4 + c^4 = d^4$ has no positive integer solutions. In fact, there are infinitely many.

For example, the solution $(a, b, c, d) = (95\,800, 217\,519, 414\,560, 422\,481)$ was discovered by Roger Frye in 1988, based on the work of Noam Elkies.

Prove that $abcd$ is divisible by 300 for any solution.

If Euler got this wrong, maybe looking for solutions is not the way to go. How can we work out what the factors are if we don't know the numbers?

All about remainders

This chapter builds on Chapter 8 of the *Dirichlet Student Notes* and Chapter 9 of the *Euler Student Notes*. It is all about the 'arithmetic of remainders'.

For example, when 63 or 18 is divided by 5, the remainder is 3. When -12 is divided by 5, the remainder is also 3. *Why?* Because 63, 18 and -12 are all 3 *more* than a multiple of 5 (namely 60, 15 and -15). We say 63, 18 and -12 are all *congruent modulo* 5, and write $63 \equiv 18 \equiv -12 \pmod 5$.

Since they have the same remainders, the difference of any pair of these numbers is a multiple of 5: $63 - 18 = 45 = 9 \times 5$; $18 - (-12) = 30 = 6 \times 5$ – *check* 63 *and* -12.

Every calculation modulo 5 can be reduced to one of 5 possible remainders: 0, 1, 2, 3 or 4. These are also called the *least residues* modulo 5.

In general, we have the following definitions.

Let a and b be any integers and m a positive integer, called the *modulus*.

We say a and b are *congruent modulo* m, written $a \equiv b \pmod m$, if a and b have the same remainder upon division by m.

Equivalently, $a \equiv b \pmod m$ whenever m divides $a - b$.

The *least residues* modulo m are the m remainders $0, 1, 2, \ldots, m - 1$.

The three most basic operations of addition, subtraction and multiplication all work seemlessly. (We'll recap division later.) You can replace a number with any other congruent number at any time, as many times as you like. So there is often more than one way to reach an answer.